WHAT IS MATHEMATICS?

Some people define mathematics to be the study of numbers and shapes. While this is largely true especially at the dawn of human civilisation when man began to invent ways of counting and keeping record of their possessions or studying shapes to help them in construction, mathematics today has departed from the very same spirit that first brought about its existence. Present-day mathematics has gone beyond the quest to answer the basic human needs as man become more sophisticated, as civilisation progresses and as technology advances. Having answered the basic needs, man’s higher intellectual need must now be satisfied. This is essentially what distinguishes man from the rest of the animals in the animal kingdom. Mathematics is no longer studied merely for its utility purposes but is an intellectual endeavour of the human spirit. It is a human endeavour that transcends the confines of the physical world. It shows the vastness that the human mind can embrace without constraints and the tremendous liberty with which the human mind conceives ideas unrestrained. It is perhaps the highest expression of the innate desire and ability in man to inquire, invent and create. The famous philosopher of the 19th century Francis Bacon once remarked:

“The science of pure mathematics, in its modern developments, may claim to be the most original creation of the human spirit.”

The mathematical landscape has indeed undergone an immense transformation. New mathematics are created constantly which further deepens man’s understanding of the intricate fabric of logical reasoning and at the same time bringing mathematics to yet another higher level of sophistication and abstraction. It is like taking a helicopter that ascends steadily thus revealing more and more of the surrounding landscape. Mathematics, in today’s context, is the systematic and logical study of, not just numbers and shapes, but any entity, be it concrete or abstract. It seeks to invent new entities, discover their properties and draw relationships amongst these entities.

Mathematics is built on Axioms – Mathematical statements that are so fundamental and ‘obvious’ that they are readily accepted as truths without ever requiring formal proofs. In fact, most of these axioms are so simple and rudimentary that they cannot be proved since there are no simpler results on which the proofs can be based. Simple as they may seem, these axioms are used to construct deep theories in mathematics. They form the foundation of the entire mathematical structure. For example, in plane geometry, one
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axiom states that a unique straight line can be drawn between any two distinct points. Also, the shortest distance between two distinct points is the length of the line segment connecting these two points. These results are elementary and perfectly commonsensical to anyone on the streets. But based on these simple axioms, a super structure called ‘Plane Geometry’ is raised which contains a myriad of deep and fascinating results. The axiom of Archimedes states that given any number, there is a number less than it. In other words, the axiom says that there are no ‘gaps’ in the real line. This is equivalent to the axiom of completeness which in essence says that the real line is complete without any ‘missing numbers’. The entire framework of Analysis is built on this elementary axiom. Other axioms include the axiom of choice and the axioms of probability.

Broadly speaking, mathematics can be divided into 5 large domains:

1. Algebra – The study of structures, systems and patterns. Groups, rings and fields are entities that belong to the domain of abstract algebra. Linear algebra deals with linear equations, matrices, determinants and vector spaces.
2. Geometry – The study of shapes and their properties. Plane geometry, topology, differential geometry and Riemannian geometry belong to this branch of mathematics.
3. Analysis – The study of continuous quantities, limits, sequences and series, and the infinite processes applied to them. It grows largely out of the calculus developed by Newton and Leibniz and was further developed by Euler, Cauchy, Weierstrass, Riemann, Lebesgues, Borel and later mathematicians. Historically, differential equations arose out of analysis. Analytic geometry is the product of the marriage between geometry and analysis. Analysis can be divided into 3 main classes: real analysis, complex analysis and functional analysis.
5. Probability – The study of random events and chance. It is also called axiomatic probability as it is based on a set of well-defined axioms. Measure theory, an offshoot of analysis has been applied to probability theory with tremendous success. Lebesgues, Borel, Kolmogorov, Feller, Doob, Cramer and Polya are some mathematicians who were responsible for placing probability on a rigorous platform. The science of decision-making and management of resources, also called operations research, rely heavily on probability theory. Stochastic processes is one branch of probability theory which deals with random variables that depend on some parameter and usually represent time. It has widespread applications in inventory modelling, transport systems and bio-medical science. Statistics is another branch of mathematics built on probability theory that has wide applications.

Is mathematics an art or is it a science? This is a question frequently asked. Mathematics, the way I see it, is both an art and a science. It is an art because there is an aesthetic side to it. There are men who pursue mathematics purely for its own sake. These are called pure mathematicians. They derive from doing mathematics great joy and satisfaction. They see mathematics as beauty, truth, structure and architecture, for so it is according to the Hungarian mathematician Paul Halmos. The British pure mathematician G.H Hardy was absolutely convinced that mathematics only has beauty and that there is certainly no

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place for ugly mathematics. The great 19th century British philosopher and mathematician Bertrand Russell very aptly presented the artistic side of mathematics when he said:

“Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.”

Mathematical arguments are elegant and exude beauty like a piece of beautiful art. Just as an artist who is able to appreciate the beauty of a piece of art, so the mathematician is able to behold the beauty of mathematics. To the experts who are able to decipher its beauty, it is absolutely wholesome, perfect, pure, mesmerising and completely satisfying to the soul. A mathematician once said that life is good for two things – discovering mathematics and doing mathematics.

To add to the above, the great mathematician Sir Michael Atiyah, a field medallist and a former president of the Royal Society made this comment when he visited Singapore. He said (I heard him with my own ears):

“If you ask me to choose between truth and beauty of mathematics, I would choose beauty.”

Mathematics is also a science because it is being applied to solve physical problems. Applied mathematicians are basically interested in using mathematics to solve practical problems. In fact, many branches of mathematics precipitated when man tried to develop mathematical techniques to solve physical problems, the calculus being one such classic example. It was developed by the 17th century scientist Sir Isaac Newton (1642 – 1727) in order to provide him the techniques to tackle problems in fluid dynamics – The method of ‘Fluxions’ as he would call it. The physical universe is so effectively explored and understood with mathematics that the great Italian astronomer and mathematician Galileo Galilei uttered that mathematics is the language of the universe. Indeed, mathematics is the vehicle on which science must advance. It is interesting to note that abstract mathematical relationships, seemingly outside the realm of physical reality, are the key to large classes of physical phenomena. This caused the greatest physicist of all time Sir Albert Einstein to make the following statement:

“How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality.”

Mathematics, a product of the human mind, is very often the key that unlocks the mystery of the working universe. It is therefore no contradiction to say that in our most theoretical moods, we may be nearest to our most practical applications. Many branches of mathematics thought to be useless have now found tremendous applications in the real world. For instance, number theory, originally pursued for its own sake without the slightest inkling of its utility value, is now being applied to cryptography and security systems with great success. When the idea of the imaginary number $i (=\sqrt{-1})$ was first conceived, it was treated with suspicion. Nobody thought it made any sense, much less having any utility value. But today, complex analysis, based on the concept of the very
same imaginary number $i$ that was frowned upon, has widespread applications in the real world. Group theory, a branch of abstract algebra plays an important role in helping scientists understand atomic structures and patterns. Functional analysis and the theory of linear operators have given physicists the exact tools they need in quantum mechanics which in turn helps in the understanding of the intricate workings of the subatomic world. Probability theory is an indispensable tool in the science of prediction and decision-making. Differential equations are widely used to model after many physical processes in the real world. Fluid mechanics, which uses many branches of mathematics like ordinary and partial differential equations, integral equation, vector and complex analysis, helps scientists to design ships, airplanes, spacecrafts and rockets. Indeed time and space would fail me to talk about the many other applications of mathematics which are so important to us.

It is absolutely right to say that mathematics and technological advancement must go hand in hand. According to the French general Napoleon, the prosperity of a state is intimately linked to the progress of mathematics. In fact, he started a school that trained people in science and mathematics in order to augment the progress of the state. Without mathematics, it is unthinkable that any form of progress would be possible since technology rides upon the wheels of mathematics. The design and functioning of modern appliances, motor cars, ships, airplanes, computers, cameras, you name them, all rely on mathematics. Mathematics is an indispensable tool in all sorts of engineering, space exploration, architecture, media industry, bio-medical science, business and finance, economics and even the social sciences. The important role that mathematics plays in our modern society can never be overemphasised.