Chapter 0

Review of Basic Concepts

Objectives:

This is an independent learning unit. It is a review of basic concepts you have learnt at 'O' level which are important prerequisites for the 'A' level course. You are expected to study this carefully on your own and work on the given assignment which is to be submitted to your tutor.

0.1 Review of the Real Number System

1. The set of **integers**, denoted by $\mathbb{Z}$, is given by $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$.

3. The set of **positive integers**, denoted by $\mathbb{Z}^+$, is given by $\mathbb{Z}^+ = \{1, 2, 3, 4, ...\}$.

4. The set of **negative integers**, denoted by $\mathbb{Z}^-$, is given by $\mathbb{Z}^- = \{-1, -2, -3, -4, ...\}$.

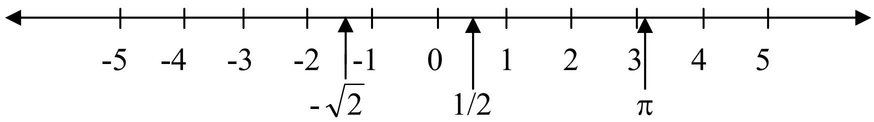
5. The set of **rational numbers**, denoted by $\mathbb{Q}$, is given by $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$. That is, a rational number is one which can be expressed as a fraction. Some examples of rational numbers are ±3, ±1, $\pm\frac{5}{4}$, $\pm\frac{1}{2}$ and 0.

   Remarks: 1. If $q = 1$ in particular, then the above set reduces to $\{p: p \in \mathbb{Z}\}$ which is simply the set $\mathbb{Z}$. This implies that $\mathbb{Z} \subset \mathbb{Q}$.

   2. The condition $q \neq 0$ in the set $\mathbb{Q}$ is necessary to ensure that the number '$\frac{p}{q}$' is always defined.

6. The set of **irrational numbers**, denoted by $\mathbb{Q}'$, is the set of numbers which are not rational. That is, they are numbers which *cannot be written* in the form $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$. Some examples of irrational numbers are $\pm\sqrt{2}$, $\pm\pi$, $\pm\log_2 3$.

7. The set of **real numbers**, denoted by $\mathbb{R}$, is the set of numbers which comprises both the rational and irrational numbers. In set notation, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
   Real numbers can be represented as points on a straight line, called the **Real Line**.



   Q: Is '∞' a real number? Why?

8. The set of **positive real numbers**, denoted by $\mathbb{R}^+$, is the set $\{x \in \mathbb{R} : x > 0\}$. Note that the number '0' is excluded as it is not a positive real number. In fact, '0' is the only real number that is neither positive nor negative.

9. The set of **positive real numbers and zero**, denoted by $\mathbb{R}_0^+$, is the set $\{x \in \mathbb{R} : x \geq 0\}$.

Self-Review 0.1: Write down the set of negative rational numbers using proper set notation.

0.2 Review of Basic Algebra

0.2.1 Review of Basic Laws of Algebra

1. For all real numbers x, y, z, $z \neq 0$, $\dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}$.

   Q: Is $\dfrac{x}{y+z} = \dfrac{x}{y} + \dfrac{x}{z}$ for all real numbers x, y, z?

2. For all real numbers x, y, $(x + y)^2 = x^2 + 2xy + y^2$. Note that in general, $(x + y)^2 \neq x^2 + y^2$.
   Q: Under what circumstances will $(x + y)^2 = x^2 + y^2$?

3. $\sqrt{(xy)} = \left(\sqrt{x}\right)\left(\sqrt{y}\right)$ for all $x, y \in \mathbb{R}_0^+$.
   Q: Is $\sqrt{(x+y)} = \sqrt{x} + \sqrt{y}$ for all $x, y \in \mathbb{R}_0^+$? When will equality hold?

4. $\left(\sqrt{x}\right)^2 = x$ for all $x \in \mathbb{R}_0^+$.

5. The modulus function, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

6. $|x|^2 = x^2$ for all $x \in \mathbb{R}$.

7. $\sqrt{x^2} = |x|$ for all $x \in \mathbb{R}$.

Self-Review 0.2: The statement '$\dfrac{x+y}{x+z} = \dfrac{y}{z}$' is *false* in general. Find conditions that must be satisfied by x, y and z in order for the statement to hold.

0.2.2 Review of the Remainder and Factor Theorems

If a **polynomial** $p(x)$ is divided by the linear factor $(ax + b)$, then we can write

$$p(x) = (ax + b)q(x) + r$$

where $q(x)$ is another polynomial of **degree** one less than that of $p(x)$ [The degree of a polynomial is defined to be the highest power of x] called the **quotient** and r is a constant called the **remainder**.

Putting $x = -\dfrac{b}{a}$ into the above gives

$p\left(-\dfrac{b}{a}\right) = \left[a\left(-\dfrac{b}{a}\right) + b\right]q\left(-\dfrac{b}{a}\right) + r \;\Rightarrow\; r = p\left(-\dfrac{b}{a}\right)$. This is called the **remainder theorem**.

In particular, if $p\left(-\dfrac{b}{a}\right) = 0$, then $r = 0$. Thus $p(x) = (ax + b)q(x)$ which implies that $(ax + b)$ is a factor of $p(x)$. This is called the **factor theorem**.

Example 0.1

Find the value of the constant k such that $(x + 1)$ is a factor of $x^3 + kx^2 - 5x - 6$.
Hence factorise $x^3 + kx^2 - 5x - 6$ completely.

Solution:

Let $p(x) = x^3 + kx^2 - 5x - 6$. Since $(x + 1)$ is a factor of $p(x)$, by the factor theorem,
$p(-1) = 0 \Rightarrow (-1)^3 + k(-1)^2 - 5(-1) - 6 = 0 \Rightarrow k = 2$.
Thus $p(x) = x^3 + 2x^2 - 5x - 6 = (x + 1)(ax^2 + bx + c)$.
Equating coefficients of x^3, $a = 1$,
Equating constants, $c = -6$,
Equating coefficient of x^2, $b + a = 2 \Rightarrow b = 1$.
Hence $x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6) = (x + 1)(x - 2)(x + 3)$ by further factorisation.

Self-Review 0.3: Solve the equation $2x^3 + x^2 - 5x + 2 = 0$ without the calculator [Ans: $\frac{1}{2}, -2, 1$]

0.2.3 Review of Quadratic Functions & Equations

The expression $f(x) = ax^2 + bx + c$ where x is a real variable, a, b and c are real constants and $a \neq 0$, is called a **quadratic function**. It is a **polynomial of degree 2** as the highest power of x is 2. We can always rewrite $f(x)$ in the form $a(x + p)^2 + q$ by **completing the square**.

Example 0.2

Complete the square for $2x^2 - x + 3$. Hence find the value of x for which the given quadratic expression takes the minimum value and write down this minimum value.

Solution:

Make coefficient of x^2 one — *add square of half the coefficient of x*

$$2x^2 - x + 3 = 2\left(x^2 - \frac{1}{2}x\right) + 3 = 2\left[\underbrace{x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2}_{\text{A perfect square}} - \left(-\frac{1}{4}\right)^2\right] + 3 = 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16}\right] + 3$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{23}{8}.$$

Clearly, the quadratic expression takes the minimum value when $x = \frac{1}{4}$ and this minimum value is

$$2\left(\frac{1}{4} - \frac{1}{4}\right)^2 + \frac{23}{8} = \frac{23}{8}.$$

The equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is called a **quadratic equation**. By completing the square, the equation can be rewritten in the form $a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} = 0$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a} = \pm\sqrt{\left(\frac{b^2-4ac}{4a^2}\right)}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

which gives the formula for obtaining the **roots** of the quadratic equation.

Remarks:

1. The nature of the roots of the quadratic equation depends solely on the sign of the expression '$b^2 - 4ac$' in the above formula, called the **Discriminant**. Can you see why? We summarise in the table below.

Table 0.1

Sign of Discriminant	**Nature of Roots**
positive	Real and distinct
zero	Real and equal
negative	Complex

2. We can also use the Discriminant to show that a quadratic function is always positive or negative for all real values of x. Consider the quadratic function $ax^2 + bx + c$ $(a \neq 0)$. If $a > 0$, the graph is **concave upward** (i.e u-shaped curve with a minimum point). If in addition, Discriminant < 0, then the quadratic equation $ax^2 + bx + c = 0$ cannot have any real root. This means that the graph must lie strictly above the x-axis and so we therefore conclude that $ax^2 + bx + c > 0$ for all real values of x (see Figure 0.1).

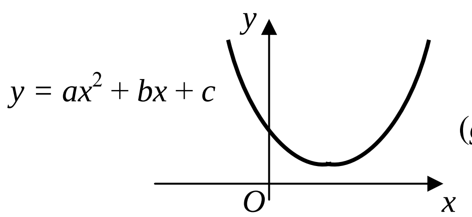


Figure 0.1
(*graph strictly above x-axis if a > 0 & Discriminant < 0*)

Similarly, if $a < 0$ and Discriminant < 0, then $ax^2 + bx + c < 0$ for all real values of x. Why?

3. The sum of roots of the quadratic equation is $-\frac{b}{a}$ and the product of roots is $\frac{c}{a}$. Can you prove this?
4. In solving a quadratic equation, we should always seek to **factorise** the quadratic expression whenever possible and the 'formula method' will only be sought if this fails.

Self-Review 0.4: Prove that $2x^2 - 3x + 2 > 0$ for all real values of x.

0.2.3 Review of Partial Fractions

Recall the following **partial fractions** decomposition:

Non-repeated linear factors: $\dfrac{px+q}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$

Repeated linear factors: $\dfrac{px^2+qx+r}{(ax+b)(cx+d)^2} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$

Non-repeated quadratic factor: $\dfrac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \dfrac{A}{ax+b} + \dfrac{Bx+C}{x^2+c^2}$

Example 0.3

Express $\dfrac{5x^2-x+2}{(2x+1)(x^2+1)}$ as a sum of partial fractions**.**

Solution:

Write $\dfrac{5x^2-x+2}{(2x+1)(x^2+1)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+1}$

$$= \frac{A(x^2+1)+(Bx+C)(2x+1)}{(2x+1)(x^2+1)}$$

Equating the numerator, $5x^2 - x + 2 = A(x^2 + 1) + (Bx + C)(2x + 1)$

Put $x = -\dfrac{1}{2}$: $5\left(-\dfrac{1}{2}\right)^2 - \left(-\dfrac{1}{2}\right) + 2 = A\left[\left(-\dfrac{1}{2}\right)^2 + 1\right] \Rightarrow A = 3$.

Equate coefficients of x^2: $5 = A + 2B \Rightarrow B = 1$.

Equate constants: $2 = A + C \Rightarrow C = -1$.

Therefore $\dfrac{5x^2-x+2}{(2x+1)(x^2+1)} = \dfrac{3}{2x+1} + \dfrac{x-1}{x^2+1}$.

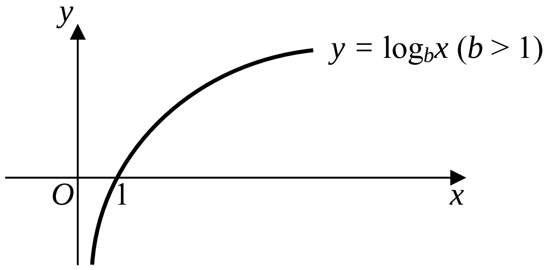
Self-Review 0.5: Show that the partial fractions of $\dfrac{7x^2-3x+2}{(3x-1)(x+1)^2}$ are $\dfrac{1}{3x-1} + \dfrac{2}{x+1} - \dfrac{3}{(x+1)^2}$.

0.3 Review of the Logarithmic and Exponential Functions

0.3.1 The Logarithmic Function

A **logarithmic function** takes the form $f(x) = \log_b x$ where $b > 0$, $b \neq 1$, $x \in R^+$. The number 'b' is called the **base** of the logarithm.
The graph of $f(x) = \log_b x$ for $b > 1$ is shown below. Observe that the graph is **strictly increasing** and it never touches the y-axis. We say that the y-axis is a **vertical asymptote** of the graph.

Figure 0.2
(*graph of f(x) = log$_b$x for b* > 1)



Q: Can you sketch the graph of $f(x) = \log_b x$ for $b < 1$?

Remarks:

1. $y = \log_b x \Leftrightarrow x = b^y$.
2. $\log_b 1 = 0$ and $\log_b b = 1$.
   To show that $\log_b 1 = 0$, we let $x = \log_b 1 \Rightarrow b^x = 1 \Rightarrow x = 0$ since $b \neq 0$. Hence $\log_b 1 = 0$.
   Q: Can you show that $\log_b b = 1$?
3. $\log_b x = \log_b y \Leftrightarrow x = y$

Recall the following 4 laws of logarithm:

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b x^a = a \log_b x$
4. $\log_b x = \dfrac{\log_c x}{\log_c b}$ (change of base law)

Example 0.4
Find x in terms of p given that $\log_p x + \log_x p^2 = 3$ where $x \neq p$.

Solution:
Firstly, we note that $\log_x p^2 = 2\log_x p = 2\left(\dfrac{\log_p p}{\log_p x}\right)$ by the change of base law.

So the equation becomes

$$\log_p x + 2\left(\frac{\log_p p}{\log_p x}\right) = 3 \Rightarrow \log_p x + \frac{2}{\log_p x} = 3.$$

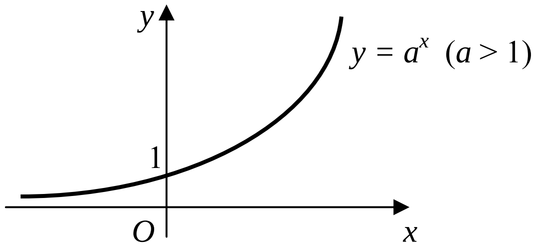
Letting $y = \log_p x$ gives $y + \dfrac{2}{y} = 3 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow (y-1)(y-2) = 0 \Rightarrow y = 1$ or 2.
Hence $\log_p x = 1$ or $2 \Rightarrow x = p$ or $x = p^2$. Since $x \neq p$, $x = p^2$.

Self-Review 0.6: Solve the equation $\log_3 2 + \log_3(x+4) = 2\log_3 x$ [Ans: $x = 4$]

0.3.2 The Exponential Function

An **exponential function** takes the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.
The graph of $f(x) = a^x$ for $a > 0$, $a \neq 1$ for $a > 1$ is shown below. Observe again that it is strictly increasing. The x-axis is a **horizontal asymptote** of the graph.

Figure 0.3
(*graph of f(x) = a^x for $a > 1$*)



Q: Can you sketch the graph of $f(x) = a^x$ for $a < 0$?

Remarks:

1. $a^x = a^y \Leftrightarrow x = y$
2. $a^0 = 1$ provided $a \neq 0$.
3. $a^{\log_a x} = x$ for $x > 0$. In particular, $e^{\ln x} = x$ for $x > 0$. Can you prove this?

Example 0.5

Find the value of x in each of the equations: (i) $e^x = 2$, (ii) $e^x = \frac{1}{2}$.

Hence, by sketching an appropriate graph, write down the solution set to the inequality $\frac{1}{2} < e^x \leq 2$.

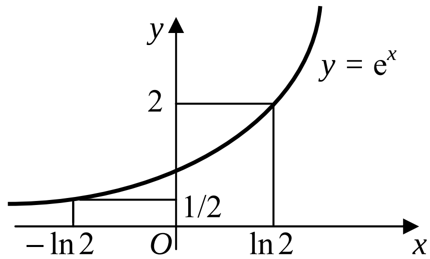
Solution:

(i) Taking ln (log to base e) both sides, $e^x = 2 \Rightarrow \ln(e^x) = \ln 2 \Rightarrow x = \ln 2$.

(ii) $e^x = \frac{1}{2} \Rightarrow \ln(e^x) = \ln\frac{1}{2} \Rightarrow x = \ln\frac{1}{2} = \ln\left(2^{-1}\right) = -\ln 2$.

By sketching the graph of $y = e^x$, we see from the sketch that $\frac{1}{2} < e^x \leq 2 \Rightarrow -\ln 2 < x \leq \ln 2$.

Figure 0.4
(*graph of y* = e^x)



The solution set to the inequality is therefore $\{x \in R: -\ln 2 < x \leq \ln 2\}$

Self-Review 0.7: Solve the equation $4^x - 3\left(2^x\right) - 4 = 0$. [Ans: $x = 2$]

0.4 Review of Basic Trigonometry

0.4.1 Definition of Trigonometric Functions and Basic Concepts

The figure below shows a circle of radius 1 unit and centre *O*.

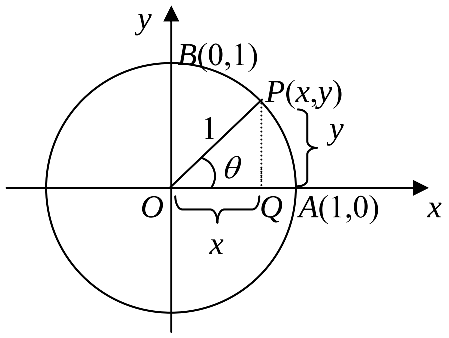


Figure 0.5
(*definition of sin θ & cos θ*)

A rotating radius *OP* rotates through an angle θ from the x-axis. Conventionally, θ is positive if it is generated in the anti-clockwise sense and is negative if generated in the clockwise sense. Let $P(x,y)$ be a general point on the circle. We define the **sine** and **cosine** of the angle θ as $x = \cos\theta$ and $y = \sin\theta$ for all $\theta \in$ R. Note that for all $\theta \in$ R, $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ since $-1 \le x \le 1$ and $-1 \le y \le 1$.

In terms of the sine and cosine functions, we may define the other four trigonometric functions:

tangent of θ, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\theta \in$ R s.t $\cos\theta \neq 0$.

cotangent of θ, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\theta \in$ R s.t $\sin\theta \neq 0$.

secant of θ, $\sec\theta = \dfrac{1}{\cos\theta}$, $\theta \in$ R s.t $\cos\theta \neq 0$.

cosecant of θ, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\theta \in$ R s.t $\sin\theta \neq 0$.

Referring to Figure 0.5 above, the three trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ are defined as the ratios between the sides of the right-angled triangle *OPQ* as follows:

$$\sin\theta = \frac{PQ}{OP} = y, \quad \cos\theta = \frac{OQ}{OP} = x, \quad \tan\theta = \frac{PQ}{OQ} = \frac{y}{x}.$$

OP is called the **hypothenus** of the right-angled triangle *OPQ*, *OQ* is called the **adjacent** side of the angle θ and *PQ* is called the **opposite** side of the angle θ.

Given any right-angled triangle *ABC* with $BC = a$, $AC = b$ and $AB = c$,

$$\sin\theta = \frac{AC}{AB} = \frac{b}{c}, \ \cos\theta = \frac{BC}{AB} = \frac{a}{c} \text{ and } \tan\theta = \frac{AC}{BC} = \frac{b}{a}.$$

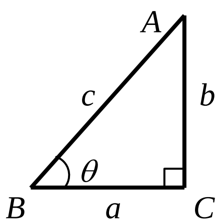


Figure 0.6
(*trigonometric ratios in a right-angled triangle*)

It is useful to memorise the exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ corresponding to certain values of θ called the **special angles**.

Table 0.2

θ	0 (0°)	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

The signs of sin θ, cos θ and tan θ are different in different quadrants.

Figure 0.7 (*the 4 quadrants*)

2nd Quadrant only sin θ +ve	1st Quadrant all three +ve
3rd Quadrant only tan θ +ve	4th Quadrant only cos θ +ve

Example 0.6

Find the exact values of sin 120°, cos 240°, tan 315°, cos(–135°) and tan($n\pi$), $n \in$ Z.

Solution:

sin 120° = sin 60° = $\frac{\sqrt{3}}{2}$; cos 240° = –cos 60° = $-\frac{1}{2}$; tan 315° = –tan 45° = –1;

cos(–135°) = –cos 45° = $-\frac{\sqrt{2}}{2}$; tan($n\pi$) = tan 0 = 0 for $n \in$ Z.

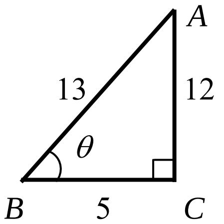
Example 0.7

Given that cos $\theta = \frac{5}{13}$ and θ is acute, find the exact values of sin θ and tan θ.

Solution:

In right-angled triangle *ABC*, *AC* = $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$ by Pythagoras' theorem.

Figure 0.8
(*Example 0.8*)

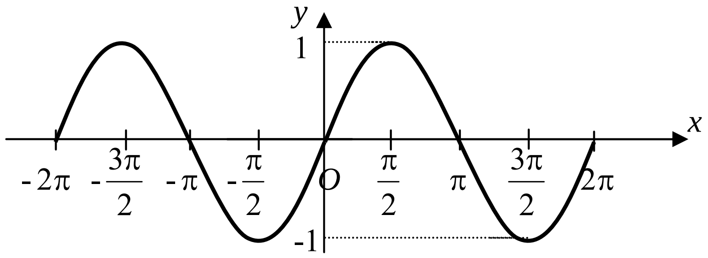


From the diagram,
sin $\theta = \frac{12}{13}$ and tan $\theta = \frac{12}{5}$.

0.4.2 Graphs of Trigonometric Functions

The Graph of $y = \sin x$

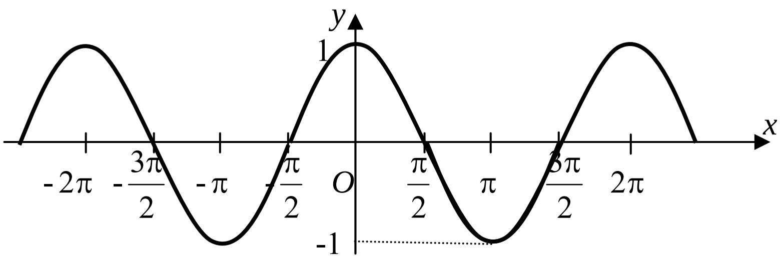
Figure 0.9
(*graph of y = sin x*)



The graph of $y = \sin x$ is **periodic** with **period** 2π. That is, $\sin x = \sin(x + 2\pi)$.

The Graph of $y = \cos x$

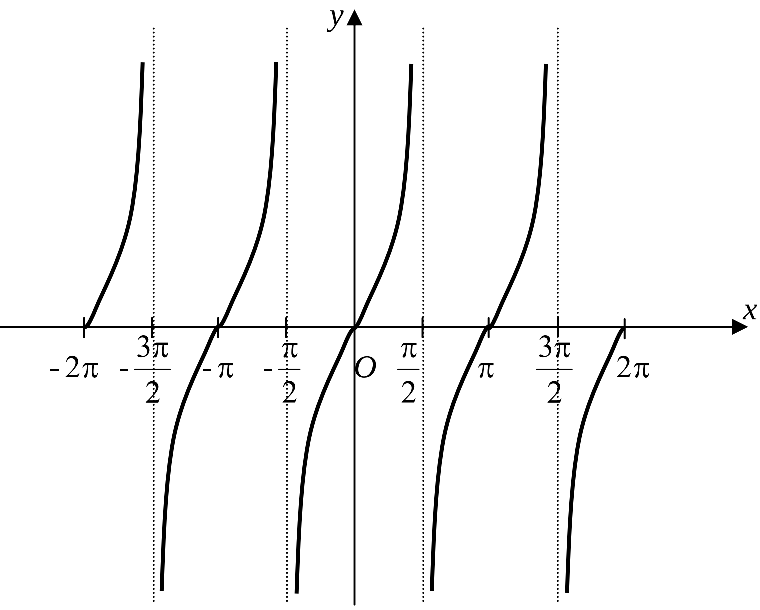
Figure 0.10
(*graph of y* = cos *x*)



The graph of $y = \cos x$ is periodic with period 2π. That is, $\cos x = \cos(x + 2\pi)$.
Note that the cosine graph can be obtained from the sine graph by shifting it to the left along the x-axis through $\frac{\pi}{2}$ radians. That is, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

The Graph of $y = \tan x$

Figure 0.11
(*graph of y = tan x*)



The graph of $y = \tan x$ is periodic with period π. That is, $\tan x = \tan(x + \pi)$.

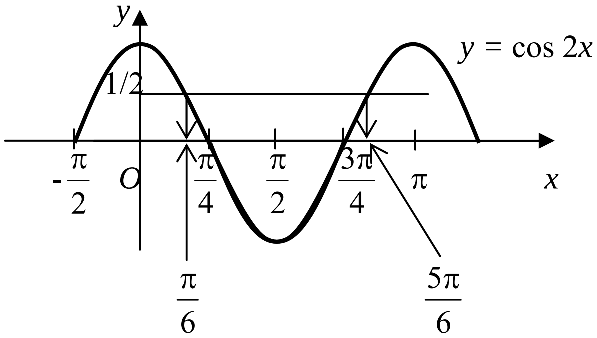
Example 0.8

Find the solution set to the inequality $\cos 2x \geq \frac{1}{2}$ for $0 < x < \pi$.

Solution:

Firstly, we solve $\cos 2x = \frac{1}{2} \Rightarrow 2x = \frac{\pi}{3}$ or $\frac{5\pi}{3} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

Figure 0.12
(*graph of* $y = \cos 2x$)



From the graph, the solution set is $\{x \in \mathrm{R}\colon\ 0 < x \leq \frac{\pi}{6}$ or $\frac{5\pi}{6} \leq x < \pi\}$.

Self-Review 0.8: Solve the inequality $\sin\left(\frac{x}{2}\right) > \frac{\sqrt{3}}{2}$. [Ans: $\frac{2\pi}{3} < x < \frac{4\pi}{3}$]

0.4.3 Trigonometric Identities

0.4.3.1 Basic Identities

1. $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$
2. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, $\tan x = \cot\left(\frac{\pi}{2} - x\right)$
3. $\sin^2 x + \cos^2 x = 1$
4. $1 + \tan^2 x = \sec^2 x$
5. $1 + \cot^2 x = \operatorname{cosec}^2 x$

0.4.3.2 The Addition Formulae

1. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
2. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
4. $\cos(x - y) = \cos x \cos y + \sin x \sin y$
5. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
6. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

0.4.3.3 The Double Angle Formulae

1. $\sin 2x = 2 \sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
3. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

0.4.3.4 The Factor Formulae

1. $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
2. $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
3. $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
4. $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

0.4.3.5 The Triple Angle Formulae

1. $\sin 3x = 3\sin x - 4\sin^3 x$
2. $\cos 3x = 4\cos^3 x - 3\cos x$

Example 0.9

Without the use of the calculator, evaluate (i) $\cos\frac{5\pi}{12}$, (ii) $\tan\frac{\pi}{8}$ exactly

Solution:

(i) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right).$

(ii) $\tan\frac{\pi}{4} = \tan 2\left(\frac{\pi}{8}\right) = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ by the double angle formula.

Since $\tan\frac{\pi}{4} = 1$, we have $\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = 1$

Letting $t = \tan\frac{\pi}{8}$ and simplifying the above equation yields $t^2 + 2t - 1 = 0$.

Solving, $t = \frac{-2 \pm \sqrt{2^2 - 4(-1)}}{2} = -1 \pm \sqrt{2}$.

Since $t = \tan\frac{\pi}{8} > 0$, $t = -1 + \sqrt{2} = \sqrt{2} - 1$.

Example 0.10

Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.
Deduce that $\sin 10°$ is a root of the cubic equation $8s^3 - 6s + 1 = 0$.

Solution:

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = (2 \sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x$
$= 2 \sin x \cos^2 x + \sin x - 2\sin^3 x = 2 \sin x (1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$.
Putting $x = 10°$ into the above identity, we have

$$\sin 30° = 3\sin 10° - 4\sin^3 10° \Rightarrow 3\sin 10° - 4\sin^3 10° = \frac{1}{2} \quad \text{since } \sin 30° = \frac{1}{2}$$
$$\Rightarrow 8 \sin^3 10° - 6 \sin 10° + 1 = 0$$
$$\Rightarrow \sin 10° \text{ is a root of the equation } 8s^3 - 6s + 1 = 0.$$

Example 0.11

Prove that $\dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$.

Solution:

$$\frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

Example 0.12

Prove that if $2 \sin(A - B) = \sin(A + B)$, then $\tan A = 3\tan B$.

Solution:

$$\sin(A - B) = \sin(A + B) \Rightarrow 2(\sin A \cos B - \cos A \sin B) = \sin A \cos B + \cos A \sin B$$
$$\Rightarrow \sin A \cos B = 3 \cos A \sin B$$
$$\Rightarrow \tan A = 3\tan B \text{ on dividing both sides by } \cos A \cos B$$

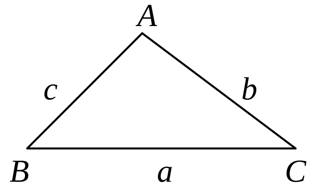
Self-Review 0.9: Prove that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$.
Deduce that $\sin\theta + \sin\phi = 2\sin\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$.

Self-Review 0.10: If $t = \tan\dfrac{\pi}{12}$, show that $t^2 + 2\sqrt{3}\,t - 1 = 0$. Deduce that $t = 2 - \sqrt{3}$.

0.4.4 The Sine and Cosine Rules

Consider triangle *ABC* below.

Figure 0.13
(*the sine and cosine rules*)



The **sine rule** states that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Q: Can you deduce from the sine rule that the largest angle is opposite the longest side?

The **cosine rule** states that
$$a^2 = b^2 + c^2 - 2bc\cos A \text{ or}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \text{ or}$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 0.13

The lengths of the sides of a triangle are 4 cm, 5 cm and 6 cm. The size of the largest angle of the triangle is θ. Calculate the value of $\cos\theta$ and hence show that $\sin\theta = \dfrac{a}{b}\sqrt{7}$ where a and b are integers.

Solution:

The largest angle is opposite the longest side. So the angle θ is opposite the side of length 6 cm.

By cosine rule, $\cos\theta = \dfrac{4^2 + 5^2 - 6^2}{2(4)(5)} = \dfrac{1}{8}$.

Using the basic identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$\sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{\left(1-\left(\frac{1}{8}\right)^2\right)} = \frac{3}{8}\sqrt{7}.$$

Self-Review 0.11: In a triangle *ABC*, *BC* = 2*x* and *AC* = *x*. Show that sin *A* = 2sin *B*. Deduce that *B* < 30°. For the case when *B* = 15°, show that $\sin A = 2\sqrt{\dfrac{2-\sqrt{3}}{4}}$.

0.4.5 Inverse Trigonometric Functions

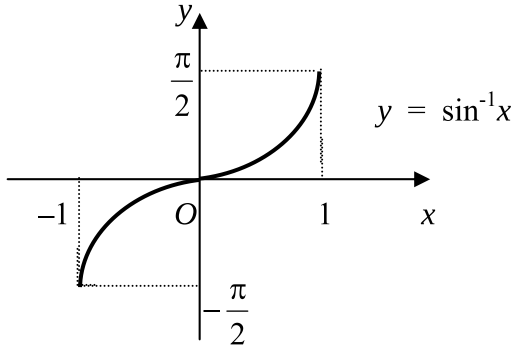
This section may be read only after you have learnt the topic on "Functions"

Consider the function $f(x) = \sin x \ (-\frac{\pi}{2} \le x \le \frac{\pi}{2})$.

The domain of f is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range of f is the interval $[-1,1]$.

Since f is 1-1, its **inverse function** f^{-1} exists and we write $f^{-1}(x) = \sin^{-1}x \ (-1 \le x \le 1)$.
The graph of $y = \sin^{-1}x$ is shown below.

Figure 0.14
(*graph of* $\sin^{-1}x$)



Note that $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$. So the range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
This is called the **principal range** of $\sin^{-1}x$.
Any value taken by $\sin^{-1}x$ in the principal range is called a **principal value**.
Unless otherwise indicated, $\sin^{-1}x$ is always the principal value.

For example, $\sin^{-1}0 = 0$, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

We can similarly define the inverse functions of $\cos x$ and $\tan x$ to be $\cos^{-1}x$ and $\tan^{-1}x$ with the understanding that principal values are always implied. We summarise our results below.

Table 0.3

$f(x)$	Domain of f	Range of f	$f^{-1}(x)$	Domain of f^{-1}	Range of f^{-1} (Principal Range)
$\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1,1]$	$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos x$	$[0,\pi]$	$[-1,1]$	$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	R	$\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The graphs of $y = \cos^{-1}x$ and $y = \tan^{-1}x$ are shown below.

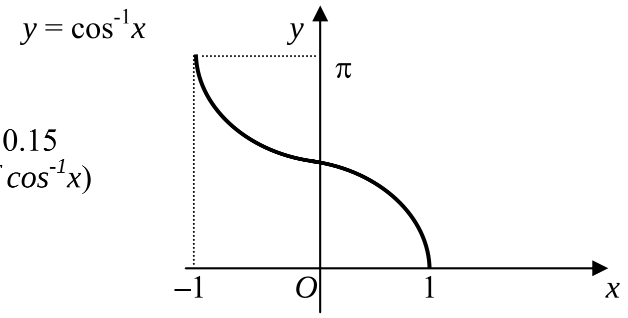


Figure 0.15
(*graph of* $\cos^{-1}x$)

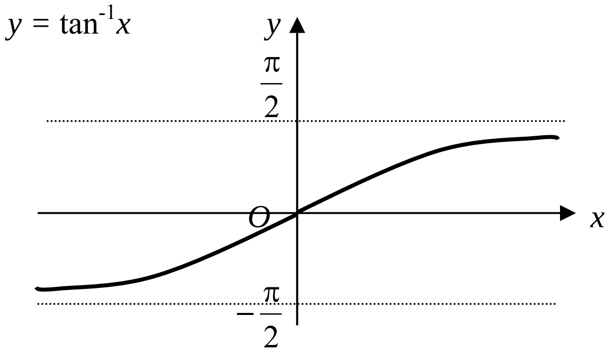


Figure 0.16
(*graph of* $\tan^{-1}x$)

Note that the graph of $y = \tan^{-1}x$ has two horizontal asymptotes $y = \pm\dfrac{\pi}{2}$.

Example 0.14

Evaluate $\cos\left(\sin^{-1}\dfrac{12}{13}\right)$ exactly without the use of the calculator.

Solution:

Let $x = \sin^{-1}\dfrac{12}{13}$. Since principal value is implied, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

We have $\sin x = \dfrac{12}{13} > 0$ and so x must be in the first quadrant in which case, $\cos x$ is positive.

Therefore $\cos x = \dfrac{5}{13}$. That is, $\cos\left(\sin^{-1}\dfrac{12}{13}\right) = \dfrac{5}{13}$.

Example 0.15

Prove that $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ where principal values are implied.

Solution:

Let $y = \operatorname{cosec}^{-1}x \Rightarrow \operatorname{cosec} y = x \Rightarrow \frac{1}{\sin y} = x \Rightarrow \sin y = \frac{1}{x} \Rightarrow y = \sin^{-1}x$.

Hence $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$.

Example 0.16

Prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ where principal values are implied.

Hence or otherwise, solve the equation $\sin^{-1}x = 2\cos^{-1}x$.

Solution:

Let $a = \sin^{-1}x$ and $b = \cos^{-1}x$. Then $\sin a = x$ and $\cos b = x$.

We have $\sin(a + b) = \sin a \cos b + \cos a \sin b = x^2 + \left(\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right) = x^2 + 1 - x^2 = 1$.

But $-\frac{\pi}{2} \le a \le \frac{\pi}{2}$ and $0 \le b \le \pi \Rightarrow -\frac{\pi}{2} \le a + b \le \frac{3\pi}{2}$ on adding. Hence $a + b = \frac{\pi}{2}$.

$\sin^{-1}x = 2\cos^{-1}x \Rightarrow \frac{\pi}{2} - \cos^{-1}x = 2\cos^{-1}x \Rightarrow \cos^{-1}x = \frac{\pi}{6} \Rightarrow x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Self-Review 0.12: (a) Show that $\sin\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{2}$, (b) Prove that $\cos^{-1}(-x) = \pi - \cos^{-1}x$.